Transcendence measures via the Thue–Siegel–Roth–Schmidt method

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(Joint work with Yann Bugeaud)
Irrationality measures

When a real number $\xi$ is proved to be irrational thanks to Diophantine approximation, the proof usually provides an infinite sequence of rationals $\left(\frac{p_n}{q_n}\right)_{n \geq 1}$ converging fast enough to $\xi$, say $0 < |q_n \xi - p_n| < \varepsilon_n$, for a vanishing sequence of real numbers $\left(\varepsilon_n\right)_{n \geq 1}$.

When it is possible to control the growth of $q_n$ and of $\varepsilon_n$, this also provides an irrationality measure for $\xi$, in the sense it is possible to find a function $f$ taking positive values and such that $|\xi - \frac{p}{q}| > f(q)$, for every rational number $\frac{p}{q}$.

This just relies on a classical trick with triangular inequalities. Moreover, in the case where the following conditions are satisfied:

(i) $\varepsilon_n < q - \varepsilon_n$,

(ii) $\limsup_{n \to \infty} \log q_n + 1/\log q_n < +\infty$,

it is possible to bound $\mu(\xi)$, the irrationality exponent of $\xi$.

Recall that: $\mu(\xi) := \sup \bar{\rho} > 0$, such that $|\xi - \frac{p}{q}| < q - \rho$ has infinitely many solutions $\bar{\rho}$.

This way, it is for instance possible to bound from above $\mu(\zeta(2))$ and $\mu(\zeta(3))$. 
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$$\mu(\xi) := \sup_{\bar{\rho} > 0} \text{such that } |\xi - p/q| < q - \rho \text{ has infinitely many solutions} \ \bar{\rho}.$$
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Transcendence measures and Mahler’s classification

We recall now the Mahler classification of real numbers. For any integer $n \geq 1$, let $w_n(\xi)$ denote the supremum of the exponents $w$ for which $0 < |P(\xi)| < H(P)^{-w}$ has infinitely many solutions in integer polynomials $P(X)$ of degree at most $n$.

Set $w(\xi) := \limsup_{n \to \infty} (w_n(\xi)/n)$.

Then, $\xi$ is an• A-number, if $w(\xi) = 0$;
• S-number, if $0 < w(\xi) < \infty$;
• T-number, if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for every integer $n \geq 1$;
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We recall some classical facts about Mahler’s classification:
• Two numbers that belong to two different classes are algebraically independent;
• Almost all real numbers are S-numbers;
• Algebraic numbers correspond to A-numbers;
• Liouville’s numbers correspond to U-numbers of degree one (the degree of a U-number is the smallest integer $n$ for which $w_n$ is infinite);
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Then,
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This lecture is motivated by the following question asked by Waldschmidt during a seminar talk of Bugeaud at "Groupe d'Étude sur les Problèmes Diophantiens" (November 2004, Jussieu, Paris).

Question (Waldschmidt).

When a real number $\xi$ is proved to be transcendantal thanks to the Thue–Siegel–Roth–Schmidt method, is it true that one can always derive from the proof a transcendance measure (possibly bad) for $\xi$?

The aim of this talk is to explain that the answer is "essentially yes".
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The case of Roth’s theorem

A fundamental result produced by the Thue–Siegel–Roth–Schmidt method is of course Roth’s theorem.

**Theorem (Roth, 1955).** Let \( \xi \) be a real number and \( \delta > 0 \). Let us assume that there exists an infinite sequence of distinct rational numbers \((p_n/q_n)_{n \geq 1}\) such that
\[
|\xi - p_n/q_n| < q_n^{-2 - \delta},
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for every \( n \geq 1 \). Then, \( \xi \) is transcendental.


**Theorem (Baker, 1964).** Under the assumption of Roth’s theorem, if moreover
\[
\limsup_{n \to \infty} \log q_n + \log q_{n+1} < +\infty,
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then \( \omega_d(\xi) < e^{2c_2} \) for a constant \( c \) independent of \( d \).

In particular, \( \xi \) is either a \( S \)-number or a \( T \)-number.

The proof is rather technical (sixteen pages including seven preliminary lemmas).
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**Theorem (Roth, 1955).** Let \( \xi \) be a real number and \( \delta > 0 \). Let us assume that there exists an infinite sequence of distinct rational numbers \( (p_n/q_n)_{n \geq 1} \) such that

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|\xi - p_n/q_n| < q_n^{-2-\delta},
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for every \( n \geq 1 \). Then, \( \xi \) is transcendental.


**Theorem (Baker, 1964).** Under the assumption of Roth’s theorem, if moreover

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\limsup_{n \to \infty} \frac{\log q_{n+1}}{\log q_n} < +\infty,
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then \( w_d(\xi) < e^{cd^2} \) for a constant \( c \) independent of \( d \). In particular, \( \xi \) is either a \( S \)-number or a \( T \)-number.
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The proof is rather technical (sixteen pages including seven preliminary lemmas).
Quantitative statements and a short proof of Baker’s theorem

It is well-known that we can bound the number of solutions of Roth’s inequality.

Theorem (Evertse, 1995).

Let $\alpha$ be an algebraic number of degree $d$ and $\delta > 0$. Then, the inequality

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has at most $c_\delta \log d \log \log d$ solutions with $q > H(\alpha)$, where $c_\delta$ only depends on $\delta$.

New idea.

Let $\xi$ be a real number such that

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Let $\alpha$ is an algebraic number of degree $d \geq 2$ such that $q_n^0 < H(\alpha) \leq q_n^0 + 1$ and $|\xi - \alpha| < H(\alpha) - \chi$, for a real number $\chi$.

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By this way, we can improve Baker’s result as follows.

Theorem (A. & Bugeaud, 2006).

Under the assumption of Baker’s theorem, we have

$$w_d(\xi) < c_1 d c_2 \log \log d$$

for some constants $c_1$ and $c_2$ both independent of $d$.

It is interesting to note that if we replace the bound of Evertse by the one of Davenport and Roth in H. Davenport & K. F. Roth,

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Quantitative statements and Schmidt Subspace Theorem

Theorem (W. M. Schmidt). Let \( m \geq 2 \) be an integer and \( \delta > 0 \). Let \( L_1, \ldots, L_m \) be linearly independent linear forms in \( x = (x_1, \ldots, x_m) \) with algebraic coefficients. Then, the set of solutions \( x = (x_1, \ldots, x_m) \in \mathbb{Z}^m \) to the inequality

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There exist extensions of this result to number fields and to \( p \)-adic valuations (see Evertse & Sclickewei, 2002).
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A proof of transcendence via the Subspace Theorem

To prove that a real number $\xi$ is transcendental, you first need linear forms with coefficients in the field $\mathbb{Q}(\xi)$ and infinitely many integer points $(x_n)_{n \geq 1}$ such that

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You assume now that $\xi$ is algebraic, so that $\mathbb{Q}(\xi) = \mathbb{Q}$, and you argue by contradiction. Since $\xi$ is algebraic, you can apply the Subspace Theorem and thanks to the pigeonhole principle you know that infinitely many of the points $x_n$ lie in a same subspace. Then:

(i) either it gives you a contradiction (take for instance a suitable limit and find that $\xi$ lies in a very special subset of $\mathbb{Q}$ such as a given number field);

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Then

$$|L_1'(x_n) \ldots L_m'(x_n)| \leq H(x_n)^{-\delta}$$

for a finite but large number $M_1$ of points $x_n$ (because $\alpha$ is very close to $\xi$).

Since the new linear forms have algebraic coefficients, you can apply the quantitative Subspace Theorem.
How to get a transcendence measure in such a case?

You assume that $\limsup_{n \to \infty} \frac{\log H(x_{n+1})}{\log H(x_n)} < +\infty$.

Then, you consider an algebraic number $\alpha$ of degree $d$ (large enough so that $\alpha$ does not lie in the very special subset of $\overline{\mathbb{Q}}$) such that, for a too large number $\chi$,

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Hence, \( \chi \) cannot be too large and that’s it.
Example: approximation by algebraic numbers of bounded degree

Theorem

(W. M. Schmidt)

Let \( \xi \) be a real number and \( \delta > 0 \). Let us assume that there exists an infinite sequence of distinct algebraic numbers \((\alpha_n)_{n \geq 1}\) of degree at most \( r \) and such that

\[
|\xi - \alpha_n| < H(\alpha_n) - r - 1 - \delta,
\]

for every \( n \geq 1 \). Then, \( \xi \) is transcendental.

Note that, contrary to the case of rational approximation, this is a difficult open problem to bound the number of solutions of inequality

\[
|\alpha - \alpha_n| < H(\alpha_n) - r - 1 - \delta,
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when \( \alpha \) is an algebraic number.

However, we can still generalize Baker's theorem as follows.

Theorem (A. & Bugeaud, 2006)

We conserve the assumption of Schmidt's theorem and we assume that

\[
\limsup_{n \to \infty} \frac{\log H(\alpha_{n+1})}{\log H(\alpha_n)} < +\infty.
\]

Then,

\[
\omega(\xi) < c_1 d c_2 \left( \log d \right)^{r-1} \left( \log \log d \right)^{r-1}
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for some constants \( c_1 \) and \( c_2 \) both independent of \( d \). In particular, \( \xi \) is either a \( S \)-number or a \( T \)-number.
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Infinite words and complexity

The complexity function of a sequence \( a = (a_n)_{n \geq 1} \) taking its values in a finite set \( A \) is the function \( n \mapsto p(n, a) \) defined by:

\[
p(n, a) = \text{Card} \{ (a_j, a_{j+1}, \ldots, a_{j+n-1}) \mid j \geq 1 \}.
\]

Clearly, the function \( p \) is non-decreasing and \( 1 \leq p(n, a) \leq (\text{Card } A)^n, n \geq 1 \).

It is extensively studied in combinatorics on words and symbolic dynamics. In particular, the entropy of a sequence (which is nothing else than the topological entropy of the underlying dynamical system) is defined as:

\[
h(a) = \lim_{n \to \infty} \frac{1}{n} \log p(n, a).
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Theorem (Morse & Hedlund, 1940).

If a sequence is eventually periodic, then \( p(n, a) \) is bounded, otherwise \( p(n, a) \) is increasing and thus \( p(n, a) \geq n + 1 \).

Moreover, there exist sequences with \( p(n) = n + 1 \) for every \( n \geq 1 \). These are Sturmian sequences.
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Moreover, there exist sequences with \( p(n) = n + 1 \) for every \( n \geq 1 \). These are Sturmian sequences.
A real number is \textit{normal} in base \( b \) if all the \( b^n \) blocks of digits of length \( n \) occur in its \( b \)-ary expansion with the right proportion, that is, with frequency \( \frac{1}{b^n} \).

Borel’s “conjecture”, 1950.

Every algebraic irrational number is a normal number.

We define the complexity of a real number \( \xi \in (0,1) \) with respect to the base \( b \) by:

\[
p(n, \xi, b) = p(n, a),
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where \( a = (a_n) \) \( n \geq 1 \) denotes the \( b \)-ary expansion of \( \xi \), that is \( \xi = \sum_{n \geq 1} a_n / b^n \).

If a real number \( \xi \) is normal in base \( b \), then its complexity is maximal, that is,

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p(n, \xi, b) = b^n, \quad \forall n \geq 1.
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Using a \( p \)-adic version of the Schmidt Subspace Theorem with three linear forms, we proved:

\textbf{Theorem (A. & Bugeaud, 2004).}

Let \( b \geq 2 \) be an integer and \( \alpha \) be an algebraic irrational number. Then,

\[
\lim_{n \to \infty} \frac{p(n, \alpha, b)}{n} = +\infty.
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B. Adamczewski & Y. Bugeaud,

\textit{On the complexity of algebraic numbers I. Expansions in integer bases},

Real numbers, normality and complexity

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Using a $p$-adic version of the Schmidt Subspace Theorem with three linear forms, we proved:

**Theorem (A. & Bugeaud, 2004).** Let $b \geq 2$ be an integer and $\alpha$ be an algebraic irrational number. Then,

$$\lim_{n \to \infty} p(n, \alpha, b)/n = +\infty.$$
We say that $\xi$ is a real number with sublinear complexity (with respect to the base $b$), if
\[ p(n, \xi, b) < cn, \]
for some constant $c$. Among these numbers, we find many classical and interesting ones:

- **Rational numbers**.
- **Lacunary numbers**: $P_n \geq 1^{1/b^n}$ with $\lim\inf u_{n+1}/u_n > 1$. For instance, the Liouville number $X_n \geq 1^{1/n!}$ is a lacunary number.
- **Automatic numbers**: these are the numbers whose $b$-ary expansion can be generated by a finite automaton. Example: the Thue-Morse-Mahler number $X_n \geq 1^{a_n b^n}$, where $a_n = 1$ if the sum of the binary digits of $n$ is even and $a_n = 0$ otherwise.
- **Sturmian numbers**: these are numbers of the form $s_\theta, x = X_n \geq 1^{b \lfloor n \theta + x \rfloor}$, where $\theta > 1$ is irrational and $x \in [0, 1)$. 
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$$\sum_{n \geq 1} \frac{1}{10^n!}$$

is a lacunary number.
- **Automatic numbers**: these are the numbers whose $b$-ary expansion can be generated by a finite automaton. Example: the Thue-Morse-Mahler number

$$\sum_{n \geq 1} a_n \frac{1}{b^n},$$

where $a_n = 1$ if the sum of the binary digits of $n$ is even and $a_n = 0$ otherwise.
- **Sturmian numbers**: these are numbers of the form

$$s_{\theta, x} := \sum_{n \geq 1} \frac{1}{b^{\lfloor n\theta + x \rfloor}},$$

where $\theta > 1$ is irrational and $x \in [0, 1)$.
Transcendence measures for real numbers with sublinear complexity

Using the method described in the first part of this talk, we are able to prove the following result.

**Theorem (A. & Bugeaud, 2006).**

Let $\xi$ be a real number with sublinear complexity. Then, one of the following situations holds:

(i) $\xi$ is either a $S$-number or a $T$-number;

(ii) $\xi$ is a rational number;

(iii) $\xi$ is a Liouville number.

This theorem is not empty! Indeed, the set of real numbers with sublinear complexity contains:

- all the rational numbers,
- some Liouville numbers (for instance, $P_n \geq \frac{1}{10^n}$),
- some $S$-numbers (for instance, $P_n \geq \frac{1}{2^n}$),

so that it is difficult to improve the theorem above. Only $T$-numbers could possibly be removed from assertion (i).

**Question.** Is it possible to find a way to make a distinction between cases (i), (ii) and (iii)?
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Question. Is it possible to find a way to make a distinction between cases (i), (ii) and (iii)?
Repetitions in words

Given an integer \( k \geq 1 \) and a finite word \( V \), we write \( V^k \) for the word \( VV \ldots V \) (\( k \) times repeated concatenation of \( V \)).

Example. The pattern \( 012012012 = (012)^3 \) is called a repetition of order 3 or simply a cube.

More generally, we can consider real repetitions. For any positive real number \( w \), we denote by \( V^w \) the word \( V^\lfloor w \rfloor V' \), where \( V' \) is the prefix of \( V \) of length \( \lceil (w - \lfloor w \rfloor) |V| \rceil \). Here, \( \lceil y \rceil \) denotes the smallest integer greater than, or equal to \( y \).

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The diophantine exponent of an infinite word

We say that an infinite word $a = a_1 a_2 \ldots$ satisfies the condition $(\ast)$ if there exist two sequences of finite words $(U_n)_{n \geq 1}$ and $(V_n)_{n \geq 1}$, and a sequence of positive real numbers $(w_n)_{n \geq 1}$ such that:

(i) $U_n V_{w_n} n$ is a prefix of $a$;

(ii) $|U_n V_{w_n} n|/|U_n V_n n| \geq \rho$;

(iii) the sequence $(|V_{w_n} n|)_{n \geq 1}$ is increasing.

The diophantine exponent of $a$, denoted by $dio(a)$, is defined as the supremum of the real numbers $\rho$ such that $a$ satisfies the condition $(\ast)$. Thus, $1 \leq dio(a) \leq +\infty$.

It is easy to show that if $a$ is eventually periodic then $dio(a) = +\infty$.

This Diophantine exponent is a measure of the periodicity of a sequence. It is first introduced in B. Adamczewski & Y. Bugeaud, Dynamics for $\beta$-shifts and Diophantine approximation, Ergod. Th. & Dynam. Sys., to appear, although it already appears under the lines in B. Adamczewski & J. Cassaigne, On Diophantine properties of real numbers generated by finite automata, Compositio Math. 142 (2006), 1351–1372.
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Repetitions and Diophantine approximation

If the \( b \)-ary expansion of a real number \( \xi \) begins with the repetitive pattern \( 0.UVw \), then, \( \xi \) is close to the rational number \( p/q := 0.UVw...Vw...Vw... \).

More precisely, \( \xi - p/q < 1/b |UVw| \) while \( q \leq b |U| \left( b |V| - 1 \right) < b |UV| \).

Thus, \( \xi - p/q < 1/q |UVw|/|UV| \).
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Thus,

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Diophantine exponent and Liouville numbers

If $\xi$ is an irrational number, we thus have

$$\mu(\xi) \geq \dio(\xi, b),$$

where $\dio(\xi, b)$ denotes the diophantine exponent of the

$b$-ary expansion of $\xi$.

Using the method introduced in

B. Adamczewski & J. Cassaigne,
On Diophantine properties of real numbers
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we prove:

Theorem (A. & Bugeaud, 2006).

Let $\xi$ be an irrational number and

$b \geq 2$ be an

integer. Let us assume that there exists a positive number $c$

such that

$$p(n, \xi, b) < cn, \forall n \geq 1.$$  

Then,

$$\max\{2, \dio(\xi, b)\} \leq \mu(\xi) \leq (2c + 1)^3(\dio(\xi, b) + 1).$$

Corollary.

Let $\xi$ be an irrational number with sublinear complexity with respect to the

base $b$, then

$\xi$ is a Liouville number if and only if

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Diophantine exponent and Liouville numbers

If $\xi$ is an irrational number, we thus have

$$\mu(\xi) \geq \text{dio}(\xi, b),$$

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Using the method introduced in B. Adamczewski & J. Cassaigne, On Diophantine properties of real numbers generated by finite automata, Compositio Math. 142 (2006), 1351–1372. we prove:

**Theorem (A. & Bugeaud, 2006).**

Let $\xi$ be an irrational number and $b \geq 2$ be an integer. Let us assume that there exists a positive number $c$ such that $p(n, \xi, b) < cn$, $\forall n \geq 1$.

Then, 

$$\max\{2, \text{dio}(\xi, b)\} \leq \mu(\xi) \leq (2c + 1)^{3(\text{dio}(\xi, b) + 1)}.$$

**Corollary.**

Let $\xi$ be an irrational number with sublinear complexity with respect to the base $b$, then $\xi$ is a Liouville number if and only if $\text{dio}(\xi, b) = +\infty$. 

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Then,

$$\max\{2, \text{dio}(\xi, b)\} \leq \mu(\xi) \leq (2c + 1)^3(\text{dio}(\xi, b) + 1).$$

**Corollary.** Let $\xi$ be an irrational number with sublinear complexity with respect to the base $b$, then $\xi$ is a Liouville number if and only if $\text{dio}(\xi, b) = +\infty$. 
Applications I: lacunary and automatic numbers

Lacunary numbers. Let $\xi = \sum_{n \geq 1} \frac{1}{b^n}$ be a lacunary number (that is, $\lim \inf_{n \to \infty} \frac{u_{n+1}}{u_n} > 1$). In that case, the Diophantine exponent can be finite or infinite. We easily get that $\xi$ is a Liouville number if $\lim \sup_{n \to \infty} \frac{u_{n+1}}{u_n} = +\infty$ and $\xi$ is either a $S$-number or a $T$-number otherwise.

Automatic numbers. Here, the Diophantine exponent is always finite as obtained in the proof of the following result:

**Theorem** (A. & Cassaigne, 2006). A Liouville number cannot be generated by a finite automaton. The latter result confirms a conjecture of Shallit, and consequently:

**Theorem** (A. & Bugeaud, 2006). Irrational automatic real numbers are either $S$-numbers or $T$-numbers. This is a first step towards a more general conjecture suggested by P.G. Becker.

**Conjecture.** Irrational automatic numbers are all $S$-numbers.
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Applications II: Sturmian numbers

For Sturmian numbers $s_{\theta, x}$, the Diophantine exponent can be finite or infinite.

Proposition. Let $s_{\theta, x}$ be a Sturmian number. Then, $\text{dio}(s_{\theta, x}) < +\infty$ if and only if $\theta$ has bounded partial quotients in its continued fractions expansion.

Theorem (A. & Bugeaud, 2006). Let $s_{\theta, x}$ be a Sturmian number. Then:

• $s_{\theta, x}$ is a Liouville number if $\theta$ has bounded partial quotients;

• $s_{\theta, x}$ is either a $S$-number or a $T$-number if $\theta$ has unbounded partial quotients.


Corollary. The two numbers $X_{n \geq 1} 1/b^{\lfloor n \sqrt{2} + \zeta(7) \rfloor}$ and $X_{n \geq 1} 1/b^{\lfloor ne + \pi \rfloor}$ are algebraically independent.
Applications II: Sturmian numbers

Sturmian numbers. For Sturmian numbers \( s_{\theta,x} := \sum_{n \geq 1} \frac{1}{b^{\lfloor n\theta + x \rfloor}} \), the Diophantine exponent can be finite or infinite.
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**Corollary.** The two numbers

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\sum_{n \geq 1} \frac{1}{b^{\lfloor n\sqrt{2} + \zeta(7) \rfloor}} \quad \text{and} \quad \sum_{n \geq 1} \frac{1}{b^{\lfloor ne + \pi \rfloor}}
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In the case where \( x = 0 \), we even have the following nice formula:

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In the case where $x = 0$, we even have the following nice formula:

$$\mu(s_\theta) = \text{dio}(s_\theta, b) = 1 + \limsup_{n \to \infty} [a_n, a_{n-1}, \ldots, a_1],$$

where $\theta = [a_0, a_1, a_2, \ldots]$. 

It is even possible to compute the continued fraction expansion of $s_\theta$. For example:

$$x \geq \frac{1}{2} \left\lfloor \frac{n(1+\sqrt{5})}{2} \right\rfloor = \frac{1}{2} + \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \left( \frac{1}{2} + \cdots \right) \right) + \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \cdots + \frac{1}{2} F_{n+1} \cdots$$

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\]

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It is even possible to compute the continued fraction expansion of \( s_\theta \). For example:

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\sum_{n \geq 1} \frac{1}{2 \lfloor n(1+\sqrt{5})/2 \rfloor} = \ldots + \frac{1}{2F_{n+1}}
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It is even possible to compute the continued fraction expansion of $s_\theta$. For example:

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$$

See for instance